HIGHER DIMENSIONAL CLIFFORD-SEVERI EQUALITIES

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Abstract. Let X be a smooth complex projective variety, $a\colon X \to A$ a morphism to an abelian variety such that $\mathrm{Pic}^0(A)$ injects into $\mathrm{Pic}^0(X)$ and let L be a line bundle on X; denote by $h^0_a(X, L)$ the minimum of $h^0(X, L \otimes a^*\alpha)$ for $\alpha \in \mathrm{Pic}^0(A)$. The so-called Clifford-Severi inequalities have been proven in [2] and [5]; in particular, for any L there is a lower bound for the volume given by:

$$\mathrm{vol}(L) \geq n! h^0_a(X, L),$$

and, if $K_X - L$ is pseudoeffective,

$$\mathrm{vol}(L) \geq 2n! h^0_a(X, L).$$

In this paper we characterize varieties and line bundles for which the above Clifford-Severi inequalities are equalities.

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Contents

1. Introduction and statement of results

A minimal surface S of general type and of maximal Albanese dimension satisfies the Severi inequality $K_S^2 \geq 4\chi(K_S)$ ([17]). The history of this result spans more than 70 years, from the original (incorrect) proof of Severi [19] to the complete proof given in [17] by the second named author. Then the first named author observed that, thanks to the generic vanishing theorem, the Severi inequality can be interpreted as a lower bound for the ratio of the volume of K_S

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to the general value of $h^0(S, K_S \otimes \alpha)$ for $\alpha \in \mathrm{Pic}^0(S)$, and that it makes sense to look for lower bounds of this type also for line bundles other than the canonical bundle. So in [2] he considered triples (X, a, L), where X is a smooth projective variety of dimension n, $a\colon X \to A$ is a morphism such that $\dim a(X) = n$ and $a^*\colon \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$ is injective, and L a nef line bundle on X. He defined the continuous rank $h^0_a(X, L)$ as the generic value of $h^0(X, L \otimes a^*\alpha)$ for $\alpha \in \mathrm{Pic}^0(A)$ and gave a set of "Clifford–Severi inequalities" in arbitrary dimension, i.e. lower bounds for the *slope* $\lambda(L) := \frac{\mathrm{vol}(L)}{h^0_a(X,L)}$. The name of Clifford was added because in dimension 1 these inequalities are just a continuous version of the classical Clifford's Lemma for line bundles on a curve.

Finally, in [3] we strengthened and refined significantly the lower bounds for $\lambda(L)$ given by the Clifford–Severi inequalities. At the same time we substantially streamlined the proofs. This was made possible by the introduction of two new tools, the continuous rank function and the eventual map, that are also essential for proving the results of this paper.

The most relevant Clifford-Severi inequalities are the following:

$$\mathrm{vol}(L) \geq n! h^0_a(X, L), \tag{1.1}$$

and, if $K_X - L$ is pseudoeffective,

$$\mathrm{vol}(L) \geq 2n! h^0_a(X, L). \tag{1.2}$$

Note that the Severi inequality for surfaces is a special case of (1.2).

When $h^0_a(X, L) > 0$, inequalities (1.1) and (1.2) can be rephrased as $\lambda(L) \geq n!$ and $\lambda(L) \geq 2n!$, respectively. We say that a line bundle L belongs to the first (respectively, second) *Clifford-Severi line* iff $h^0_a(X, L) > 0$ and $\lambda(L) = n!$ (respectively, $\lambda(L) = 2n!$).

Both inequalities (1.1) and (1.2) are sharp: ample line bundles L on abelian varieties $X = A$ clearly belong to the first Clifford Severi line; in addition, if $a\colon X \to A$ is a double cover of an abelian variety branched on a smooth ample divisor $B \in |2R|$, any line bundle $L = a^*N$ with $0 \leq N \leq R$ verifies $\lambda(L) = 2n!$, i.e. L belongs to the Second Clifford-Severi line.

Still, the classification of these extremal cases has proven surprisingly hard, due to the fact that the proof of the inequalities in [17] and in [2] is based on a limiting argument. For example, the classification of surfaces on the Severi line $K_S^2 = 4\chi(K_S)$ was achieved only in 2015 ([3] and [15], independently), ten years later than [17]. The higher dimensional cases and the case of a general L (even for surfaces) were completely open.

In this paper, we classify the triples (X, L, a) achieving equality in (1.1) and (1.2); the classification essentially reduces to the known examples described above. Let us state our main results.

Given a morphism $a\colon X \to A$ of a n-dimensional variety X to an abelian variety A, we say that a is *strongly generating* if $a^*\colon \operatorname{Pic}^0(A) \to \operatorname{Pic}^0(X)$ is injective, and *of maximal a-dimension* if $\dim a(X) = n$ (i.e, if a is finite onto its image).

First we give a characterization of pairs (X, L) on the first Severi line:

Theorem 1.1. *Let X be a smooth variety of dimension $n \geq 1$, let $a\colon X \to A$ be a strongly generating map to a q-dimensional abelian variety such that X is of maximal a-dimension; let $L \in \operatorname{Pic}(X)$ be a line bundle with $h^0_a(X, L) > 0$.*

If $\lambda(L) = n!$, then $q = n$ and $a\colon X \to A$ is a birational morphism.

Next we consider pairs (X, L) on the second Clifford-Severi line.

If the assumptions of Theorem 1.1 are satisfied, one can define the *eventual map* given by L ([5], cf. §2.4): this is a generically finite map whose degree we denote by m_L. For $n \geq 2$, we have the following characterization:

Theorem 1.2. *Let X be a smooth variety of dimension $n \geq 2$, let $a\colon X \to A$ be a strongly generating map to a q-dimensional abelian variety such that X is of maximal a-dimension; let $L \in \operatorname{Pic}(X)$ be a line bundle with $h^0_a(X, L) > 0$.*

If $K_X - L$ is pseudoeffective and $\lambda(L) = 2n!$, then:

(i) $q = n$, $\deg a = 2$*;*
(ii) *if $\chi(K_X) > 0$, then $K_X = a^*N + E$ for some ample N in A and some (possibly non effective) a-exceptional divisor E, $m_{K_X} = 2$ and a is birationally equivalent to the eventual map of K_X;*
(iii) *if in addition $K_X - L + \alpha$ is effective for some $\alpha \in \operatorname{Pic}^0(A)$, then $m_L = 2$ and a is birationally equivalent to the eventual map of L.*

The case of curves is dealt with separately (Theorem 3.5). It is worth observing that the above results are new in any dimension.

Remark 1.3. It is natural to ask whether one can improve the bound $\lambda(L) \geq 2n!$ if $\deg a \neq 2$ and $K_X - L$ is pseudoeffective. We have indeed proven stronger bounds in [3] under the hypotheses that $\deg a = 1$ or that a not composed with an involution. In the same paper however, we have constructed examples of varieties with Albanese morphism of arbitrarily large degree and slope arbitrarily close to $2n!$ (Example 7.8). So, the answer to the above question is negative. Note that the examples given in [3] have $\deg a = 2^k$ for some k.

Theorem 1.2 gives the following characterization of varieties on the second Clifford-Severi line for $L = K_X$, extending the analogous result for surfaces ([3], [15]).

Corollary 1.4. *Let X be a complex projective minimal variety of dimension $n \geq 2$ with terminal singularities, let $a\colon X \to A$ be the Albanese map and let $\omega_X = \mathcal{O}_X(K_X)$ be the canonical sheaf.*

If X is of maximal Albanese dimension and $K_X^n = 2n!\chi(\omega_X) > 0$, then:

(i) *$q = n$ and* $\deg a = 2$*;*
(ii) *if $X' \to X$ is a desingularization and $a'\colon X' \to A$ is the Albanese map, then $K_{X'} = a'^*N + E$ for some ample N in A and a'-exceptional divisor E, $m_{K_{X'}} = 2$ and a' is birationally equivalent to the eventual map of $K_{X'}$.*

The key point in our proofs are the new techniques introduced in [5], i.e. the construction of the eventual map associated to a line bundle such that $h^0_a(X, L) > 0$ (see 2.4), and the *continuous rank function* (see 2.2): given any ample line bundle H on A there is a natural way to define a real valued function $\phi(x) = h^0_a(X, L + xa^*H)$ of $x \in \mathbb{R}$, which is convex and continuous. With this new approach the limiting argument of [17] and [2] is embodied in the definition of the rank function. Moreover, the Clifford-Severi inequalities imply a relation between the rank function and the volume function $\mathrm{vol}_X(L + xa^*H)$ ([14]), i.e. the non-negativity of the functions $\mathrm{vol}_X(L + xa^*H) - n!h^0_a(X, L + xa^*H)$ and $\mathrm{vol}_X(L+xa^*H)-2n!h^0_a(X, L+xa^*H)$, respectively. We approach the classification problem by studying the minima of these two functions.

When $L = K_X$, we have proven (Theorem 1.2 (ii)) that K_X is the pullback of an ample divisor on A plus an a-exceptional divisor. In the case of surfaces we can say more (cf. [3]): the a-exceptional divisor is effective. This is equivalent to the fact that, given the Stein factorization of a: $X \to \overline{X} \to A$, the surface $\overline{X}$ has canonical singularities. It is thus very natural to pose the following question, which at the moment seems to be out of reach.

Question 1.5. In the hypotheses of Theorem 1.1, or of 1.2, is it true that $L = a^*N + E$ for some ample N in A and some *effective* a-exceptional divisor E?

2. Preliminary results

2.1. **Notation and conventions.** We work over the complex numbers; varieties and subvarieties are assumed to be irreducible and projective. Since our focus is on birational geometry, a *map* is a rational map and we denote all maps by solid arrows. We say that two dominant maps $f\colon X \to Z$, $f'\colon X \to Z'$ are *birationally equivalent* if there exists a birational isomorphism $h\colon Z \to Z'$ such that $f' = h \circ f$.

We do not distinguish between divisors and line bundles and use both the additive and multiplicative notation, depending on the situation.

If d is a non-negative integer, we write "$d \gg 0$" instead of "d large and divisible enough".

We recall briefly some definitions and results from [5], Sections 3 and 4. We refer the reader to [5] for more details.

2.2. **The continuous rank.** The starting point is a smooth variety X of dimension n with a map $a\colon X \to A$ to an abelian variety of dimension q. We say that a is *strongly generating* if $a^*\colon \operatorname{Pic}^0(A) \to \operatorname{Pic}^0(X)$ is injective and we say that X is *of maximal a-dimension* if $\dim a(X) = n$. We assume throughout all the section that both these conditions hold for $a\colon X \to A$; to simplify the notation, given $\alpha \in \operatorname{Pic}^0(A)$ we write α instead of $a^*\alpha$. Given a line bundle $L \in \operatorname{Pic}(X)$, we define its *continuous rank* (with respect to a) as the minimum $h^0_a(X, L)$ of $h^0(X, L \otimes \alpha)$ for $\alpha \in \operatorname{Pic}^0(A)$.

Consider the cartesian diagram:

$$\begin{array}{ccc} X^{(d)} & \xrightarrow{\widetilde{\mu_d}} & X \\ {\scriptstyle a_d}\downarrow & & \downarrow{\scriptstyle a} \\ A & \xrightarrow{\mu_d} & A \end{array} \tag{2.1}$$

where μ_d denotes multiplication by d. The map $\widetilde{\mu}_d$ is étale of degree d^{2q} and $X^{(d)}$ is connected since a is strongly generating; we set $L^{(d)} := \widetilde{\mu}_d^* L$.

One of the reasons for introducing the continuous rank is the following *multiplicative property*:

$$h^0_{a_d}(X^{(d)}, L^{(d)}) = d^{2q} h^0_a(X, L) \quad \text{for any } d \in \mathbb{N}. \tag{2.2}$$

Now fix a very ample line bundle $H \in \operatorname{Pic}^0(A)$ and set $M := a^*H$, and, more generally, $M_d := a_d^*H$ for all $d \in \mathbb{N}$. The line bundles $M^{(d)}$ and M_d on $X^{(d)}$ are related by the formula (see for example [16] pag. 75 formula (iv)):

$$M^{(d)} = d^2 M_d \qquad \operatorname{mod}\ \operatorname{Pic}^0(A). \tag{2.3}$$

For $x \in \mathbb{R}$ consider the $\mathbb{R}$-line bundle $L_x := L + xM$; if x is rational, then we have that for $d \gg 0$ (meaning d big and divisible enough, see 2.1) the line bundle $L^{(d)} + xd^2M_d$ is integral and by (2.3) it is equivalent modulo $\operatorname{Pic}^0(A)$ to $(L_x)^{(d)} = L^{(d)} + xM^{(d)}$. Thus

$$h^0_a(X, L_x) := \frac{1}{d^{2q}} h^0_{a_d}(X^{(d)}, L^{(d)} + xd^2 M_d)$$

is well defined thanks to (2.2).

The function $x \mapsto h^0_a(X, L_x)$ just defined for $x \in \mathbb{Q}$ can be extended to a continuous, convex function $\phi(x)$ on $\mathbb{R}$. In particular, ϕ has one-sided derivatives at every point $x \in \mathbb{R}$ and is differentiable except at most at countably many points.

Given a subvariety $T \subseteq X$, we define the *restricted continuous rank* $h^0_a(X_{|T}, L)$ as the generic value of $\dim \operatorname{Im}(H^0(X, L\otimes\alpha) \to H^0(T, (L\otimes\alpha)_{|T}))$, for $\alpha \in \operatorname{Pic}^0(A)$. The restricted continuous rank also satisfies the multiplicative property (2.2), so for rational values of x one can define $h^0(X_{|T}, L_x)$ as above, and also in this case the function $h^0(X_{|T}, L_x)$ extends to a continuous function of $x \in \mathbb{R}$. Using this definition we are able to give an explicit formula for the left derivative of ϕ (cf. [5, Theorem 4.2]):

$$D^-\phi(x) = \lim_{d\to\infty} \frac{1}{d^{2q-2}} h^0_{a_d}(X^{(d)}_{|M_d}, (L_x)^{(d)}), \quad \forall x \in \mathbb{R}. \tag{2.4}$$

2.3. **Slope and degree of subcanonicity.** Assume that $h^0_a(X, L) > 0$ and define the slope of L as:

$$\lambda(L) := \frac{\mathrm{vol}(L)}{h^0_a(X, L)}.$$

Note that by [5, Proposition 3.2] the line bundle L is big, and so $\lambda(L) > 0$.

The Clifford-Severi inequalities give explicit lower bounds for $\lambda(L)$, which involve the dimension of X and the so called *numerical degree of subcanonicity* of L (with respect to M). This is defined as:

$$r(L, M) := \frac{LM^{n-1}}{K_X M^{n-1}} \in (0, \infty]. \tag{2.5}$$

If $r(L, M) = \infty$, namely if $K_X M^{n-1} = 0$ then $\kappa(X) \le 0$, because M is big. On the other hand, we have $\kappa(X) \ge 0$ since X is of maximal a-dimension, so we conclude that $r(L, M) = \infty$ implies $\kappa(X) = 0$.

2.4. **The eventual map and the eventual degree.** Assume that $h^0_a(X, L) > 0$. One of the main results of [5] (Theorem 3.7) is the definition of the *eventual map* associated with L: it is a generically finite dominant map $\varphi\colon X \to Z$, characterized up to birational equivalence by the following properties:

(a) $a\colon X \to A$ factorizes as $X \xrightarrow{\varphi} Z \xrightarrow{g} A$ for some map g.

(b) By property (a), we have a cartesian diagram:

$$\begin{tikzcd}
X^{(d)} \arrow[r, "\varphi^{(d)}"] \arrow[d, "\widetilde{\mu_d}"] & Z^{(d)} \arrow[r] \arrow[d] & A \arrow[d, "\mu_d"] \\
X \arrow[r, "\varphi"] & Z \arrow[r, "g"] & A
\end{tikzcd}$$

Then the map $\varphi^{(d)}$ is birationally equivalent to the map given by $|L^{(d)} \otimes \alpha|$ for $d \gg 0$ and $\alpha \in \mathrm{Pic}^0(A)$ general.

The degree of φ is denoted by m_L and is called the *eventual degree* of L. It is immediate to see that L and $L^{(d)}$ have the same eventual degree, so for $x \in \mathbb{Q}$ we choose d such that ${L_x}^{(d)}$ is integral and we set $m_{L_x} := m_{(L_x)^{(d)}}$. The function $x \mapsto m_{L_x}$ is non increasing and takes integer values, so it can be extended to a left continuous function of $x \in \mathbb{R}$.

2.5. **Continuous resolution of the base locus.** Let $\sigma\colon \widetilde{X} \to X$ be a birational morphism, set $\widetilde{L} := \sigma^*L$ and denote by $\widetilde{a}\colon \widetilde{X} \to A$ the map induced by a; we have

$$h^0_{\widetilde{a}}(\widetilde{X}, \widetilde{L}) = h^0_a(X, L), \quad \mathrm{vol}(\widetilde{L}) = \mathrm{vol}(L)$$

and the eventual map $\widetilde{\varphi}$ given by $\widetilde{L}$ is equal to $\varphi \circ \sigma$ (in particular, $m_{\widetilde{L}} = m_L$). So, for many purposes, we may replace (X, L) by $(\widetilde{X}, \widetilde{L})$.

Since it is easier to deal with morphisms than with rational maps, the following construction (*continuous resolution of the base locus*) is often useful. It is possible (cf. [2, Sec. 3] [5, Sec. 2.4]) to choose the morphism $\sigma\colon \widetilde{X} \to X$ in such a way that there is a decomposition $\widetilde{L} = P + D$ as a sum of effective divisors, such that for $d \gg 0$:

(a) the system $|P^{(d)} \otimes \alpha|$ is free for all $\alpha \in \mathrm{Pic}^0(A)$;
(b) $|P^{(d)} \otimes \alpha|$ is the moving part of $|\widetilde{L}^{(d)} \otimes \alpha|$ for $\alpha \in \mathrm{Pic}^0(A)$ general.

Clearly, one has $h^0_{\widetilde{a}_d}(\widetilde{X}^{(d)}, P^{(d)}) = h^0_{\widetilde{a}_d}(\widetilde{X}^{(d)}, \widetilde{L}^{(d)}) = h^0_{a_d}(X^{(d)}, L^{(d)}) = d^{2q} h^0_a(X, L)$, where $\widetilde{a}\colon \widetilde{X} \to A$ is the morphism induced by a. Note that P and D are divisors on $\widetilde{X}$ and that the modification $\sigma\colon \widetilde{X} \to X$ satifying the above properties is not unique; however we will usually assume that a suitable σ has been chosen and we will call P and D the *continuous moving part* and the *continuous fixed part* of L, respectively. Actually, given $x \in \mathbb{Q}$ such that $h^0_a(X, L_x) > 0$ it is convenient to be able to speak about the continuous moving part P_x of L_x and its continuous fixed part D_x. In order to do this we choose $d \in \mathbb{N}$ such that $(L_x)^{(d)}$ is integral and then choose a modification $\eta\colon Y \to X^{(d)}$ on which $\eta^*\left((L_x)^{(d)}\right)$ decomposes as the sum $P + D$ of its continuous moving and fixed part. One would like to have divisors P_x and D_x on (a modification of) X that pull back to P and D on Y and call these the continuous moving and fixed part of L_x, but such P_x and D_x usually do not exist. Moreover the construction of P and D involves choosing first an integer $d \gg 0$ and then a suitable birational morphism $\eta\colon Y \to X^{(d)}$. However, some of the invariants attached to the decomposition do not depend on these choices and so it makes sense to define them, in particular we can define:

- $h^0_a(X, P_x) := \frac{1}{d^{2q}} h^0_\alpha((Y, P) = \frac{1}{d^{2q}} h^0_a(X, (L_x)^{(d)})$, where $\alpha\colon Y \to A$ is the map induced by a_d;
- $(P_x)^{n-1}M := \frac{1}{d^{2q}} P^{n-1}(\eta^* M)$;
- $m_{P_x} = m_P = m_{L_x}$.

Moreover, we will sometimes write $(P_x)^{(d)}$, implying that we have chosen d and $\eta\colon Y \to X^{(d)}$ as above and we are considering the continuous moving part P of $\eta^*\left((L_x)^{(d)}\right)$.

2.6. **The volume function.** We refer the reader to [12], [13, § 2.2.C] and [9] for a complete account of the properties of the volume and of the restricted volume. Here we just recall what we need in our proofs.

The *volume* of a line bundle $L \in \mathrm{Pic}(X)$ is defined as:

$$\mathrm{vol}(L) = \mathrm{vol}_X(L) := \limsup_m \frac{n! h^0(X, mL)}{m^n}$$

and, more generally, given a k-dimensional subvariety $T \subset X$ the *restricted volume* is defined as:

$$\mathrm{vol}_{X|T}(L) := \limsup_m \frac{k! h^0(X_{|T}, mL)}{m^k}.$$

For $t \in \mathbb{N}$ one has $\mathrm{vol}(tL) = t^n \mathrm{vol}(L)$ and $\mathrm{vol}_{X|T}(tL) = t^k \mathrm{vol}_{X|T}(L)$, so both definitions generalize naturally to $\mathbb{Q}$-line bundles. Below we state in our setting some useful consequences of the properties of the volume of a $\mathbb{Q}$-line bundle (cf. also [5]). We assume $h^0_a(X, L) > 0$ and we denote by M a general element of $|M| = |a^*H|$ and by M_d a general element of $|M_d| = |a_d^*H|$.

(i) if $L = P + D$, with D effective, then $\mathrm{vol}(L) \geq \mathrm{vol}(P)$ and $\mathrm{vol}_{X|M}(L) \geq \mathrm{vol}_{X|M}(P)$;

(ii) if $\eta \colon \widetilde{X} \to X$ is a birational morphism, then $\mathrm{vol}_{\widetilde{X}}(\eta^* L) = \mathrm{vol}_X(L)$ and $\mathrm{vol}_{\widetilde{X}|\widetilde{M}}(L) = \mathrm{vol}_{X|M}(L)$, where $\widetilde{M} := \eta^* M$;

(iii) $\mathrm{vol}_{X^{(d)}}(L^{(d)}) = d^{2q} \mathrm{vol}_X(L)$ and $\mathrm{vol}_{X^{(d)}|M^{(d)}}(L^{(d)}) = d^{2q} \mathrm{vol}_{X|M}(L)$ for $M \in |M|$ general;

(iv) if L is nef then $\mathrm{vol}_X(L) = L^n$ and $\mathrm{vol}_{X|M}(L) = L^{n-1}M = \mathrm{vol}_M(L_{|M})$ for $M \in |M|$ general.

The definition of volume can be extended to $\mathbb{R}$-line bundles, giving a continuous function on $N^1(X, \mathbb{R})$. We consider the continuous function $\psi(x) := \mathrm{vol}(L_x)$, for $x \in \mathbb{R}$: if $\mathrm{vol}(L_x) > 0$ (i.e., if L_x is big), ψ is differentiable and we have (cf Thm. A and Cor. C of [6], and also [14, Cor. C]):

$$\psi'(x) = n \, \mathrm{vol}_{X|M}(L_x), \tag{2.6}$$

where $\mathrm{vol}_{X|M}$ is the restricted volume.

We close this section with a remark that allows us to compare the derivatives of $\psi(x)$ and of the continuous rank function $\phi(x)$. This will be a key point of our arguments.

Remark 2.1. Fix $x \in \mathbb{Q}$ such that $h^0_a(X, L_x) > 0$ and consider the continuous moving and fixed part P_x and D_x of L_x as in Section 2.5.

Although P_x and D_x are not defined on X, thanks to the above properties (ii) and (iii) of the volume, we can nevertheless define their volume as we did in 2.5

for the continuous rank as follows:

$$\mathrm{vol}_{X|M}(P_x) := \frac{1}{d^{2q}} \mathrm{vol}_{X^{(d)}|M^{(d)}} \left((P_x)^{(d)}\right),$$

where $d \gg 0$. Note that using property (iii) we still have property (i): $\mathrm{vol}_{X|M}(L_x) \geq \mathrm{vol}_{X|M}(P_x)$. By property (iv) we have:

$$\begin{aligned} \mathrm{vol}_{X|M}(P_x) &= \frac{1}{d^{2q}} \left((P_x)^{(d)}\right)^{n-1} M^{(d)} = \frac{1}{d^{2q-2}} \left((P_x)^{(d)}\right)^{n-1} M_d = \\ &= \frac{1}{d^{2q-2}} \mathrm{vol}_{X^{(d)}|M_d} \left((P_x)^{(d)}\right). \end{aligned} \tag{2.7}$$

Assume now that the inequality $\mathrm{vol}_{X^{(d)}|M_d}\left((P_x)^{(d)}\right) \geq Ch^0_{a_d}(X^{(d)}{}_{|M_d}, (P_x)^{(d)}))$ holds for some constant $C > 0$ and all $d \gg 0$. By (2.7) we obtain

$$\mathrm{vol}_{X|M}(L_x) \geq \mathrm{vol}_{X|M}(P_x) \geq C\frac{1}{d^{2q-2}} h^0_{a_d}(X^{(d)}{}_{|M_d}, (P_x)^{(d)}) = \frac{1}{d^{2q-2}} Ch^0_{a_d}(X^{(d)}{}_{|M_d}, (L_x)^{(d)}),$$

where the first inequality follows by property (i) and the final equality comes from the very definition of P_x (see §2.5). Combining (2.4), (2.6) and (2.7), we obtain:

$$\psi'(x) \geq nCD^- \phi(x).$$

3. Proofs of the main results

This section is devoted to proving our main results; we use freely the notation introduced in Section 2.

3.1. **The first Clifford-Severi line.**

Proof of Theorem 1.1. Take H very ample on A and let $M = a^*H$. By Theorem 6.7 in [5] we have:

$$\lambda(L) \geq \frac{2r(L, M)}{2r(L, M) - 1} n! \tag{3.1}$$

where $r(L, M) = \frac{LM^{n-1}}{K_X M^{n-1}}$ is the subcanonicity index of L with respect to M (2.3). Since by hypothesis $\lambda(L) = n!$, we have that $r(L, M) = +\infty$ and therefore $\kappa(X) = 0$ (cf. § 2.3).

Now, X is of maximal Albanese dimension and hence, by a criterion of Kawamata ([11]) we have that X is birational to an abelian variety. Since $n = \dim X \leq q = \dim A$ and the map a is strongly generating, we have that $q = n$ and a is birational. □

Remark 3.1. The following analogue of (3.1) has been proved in [2] for L nef:

$$\lambda(L) \geq \frac{2r(L)}{2r(L)-1}\, n! \tag{3.2}$$

where $r(L)$ is the infimum of the r such that $rK_X - L$ is pseudoeffective. Of course $r(L, M) \leq r(L)$ for the pullback M of any very ample H on A, so (3.1) is a stronger inequality than (3.2) above. This is a significant improvement: when $\lambda(L) = n!$ inequality (3.2) tells us only that K_X is not big, while inequality (3.1) gives the much stronger condition $\kappa(X) = 0$, which is crucial for proving Theorem 1.1 and, as a consequence, Theorem 1.2.

3.2. **The second Clifford-Severi line.** As in Section 2, we fix a very ample line bundle H on A, let $M = a^*H$ and $L_x = L + xM$, for $x \in \mathbb{R}$; recall that the eventual degree m_{L_x} is defined for all x such that $h^0_a(X, L_x) > 0$.

We need the following result, which strenghtens [5, Theorem 6.9 (i)]. The argument is a slight modification of the proof of that theorem. Since that proof is spread in several intermediate results, for the ease of the reader we sketch here the main steps and refer to [5] for further details.

Proposition 3.2. *In the assumptions of Theorem 1.2, assume that we have that* $m_{L_x} = 1$ *for all* $0 \geq x \in \mathbb{Q}$ *such that* $h^0_a(X, L_x) > 0$. *Then*

$$\mathrm{vol}(L) \geq \frac{5}{2}\, n!\, h^0_a(X, L). \tag{3.3}$$

Proof. First of all observe that the hypothesis about the eventual degree is stable under multiplication maps, under passing to the continuous moving part and restricting to a general section M_d (see §2).

More concretely, if $x \in \mathbb{Q}$, $d \gg 0$ and M_d is a smooth general member of $|a_d^*(H)|$ then we have

$$m_{L_x} = 1 \;\Rightarrow\; m_{P_x} = 1 \;\Rightarrow\; m_{P_x^{(d)}} = 1 \;\Rightarrow\; m_{P_x{}^{(d)}_{|M_d}} = 1. \tag{3.4}$$

We can assume $h^0_a(X, L) > 0$, since the result is trivially true otherwise. The proof works by induction on n.

For $n = 2$ we only need to follow the chain of implications given in [5], where Proposition 5.4 (ii) implies Theorem 5.5 (ii) which finally implies Proposition 6.14 (i): $\mathrm{vol}(L) \geq 5h^0_a(X, L)$. The assumption made in [5] is that a is of degree 1. We do not have this hypothesis here; however, the only fact used is that the linear systems $|(P_x)^{(d)}|_{|M_d}$ (and hence $|(P_x)^{(d)}_{|M_d}|$) are generically injective for $d \gg 0$, and this is true by (3.4).

For the inductive step, take $0 > x \in \mathbb{Q}$ with $h^0_a(X, L_x) > 0$ and consider its continuous moving part P_x (cf. § 2.5). Note that $K_X - L_x = (K_X - L) - xM$ is pseudoeffective, since $x < 0$ and $K_X - L$ is pseudoeffective. If t is such that

$(L_x)^{(t)}$ is integral and we have chosen a continuous resolution $\eta\colon Y \to X^{(t)}$ of the continuous base locus of $(L_x)^{(t)}$, then it is easy to see that $K_{Y^{(t)}} - (P_x)^{(t)}$ also is pseudoeffective.

So, since $m_{P_x} = 1$, for $d \gg 0$ the inductive hypothesis gives:

$$\begin{aligned} \mathrm{vol}_{X^{(d)}|M_d}\left((P_x)^{(d)}\right) &= ({P_x}^{(d)})^{n-1}M_d = \mathrm{vol}_{M_d}\left({(P_x)^{(d)}}_{|M_d}\right) \geq \\ &\geq \frac{5}{2}(n-1)!h^0_{a_d}(M_d, {(P_x)^{(d)}}_{|M_d}) \geq \frac{5}{2}(n-1)!h^0_{a_d}(X^{(d)}_{|M_d}, (P_x)^{(d)}), \end{aligned} \tag{3.5}$$

where the first equalities hold since $(P_x)^{(d)}$ is nef (cf. § 2.6). As in §2.2, denote by $\phi(x) = h^0_a(X, L_x)$ the continuous rank function and by $\psi(x) = \mathrm{vol}(L_x)$ the volume function; by Remark 2.1, inequality (3.5) gives:

$$\psi'(x) \geq \frac{5}{2}n!D^-\phi(x)$$

for $0 \geq x \in \mathbb{Q}$. Since $\psi'(x)$ is continuous and $D^-\phi(x)$ is non decreasing, the same inequality holds for $x \in \mathbb{R}$. Now the desired inequality is obtained by taking the integral:

$$\mathrm{vol}(L) = \int_{-\infty}^0 \psi'(x)dx \geq \frac{5}{2}n! \int_{-\infty}^0 D^-\phi(x)dx = \frac{5}{2}n!h^0_a(X, L).$$

□

Remark 3.3. The inequality (3.3) of Proposition 3.2 is one of the new stronger Clifford–Severi inequalities proven in [5]. We do not know whether this inequality is sharp (i.e. whether there are triples (X, a, L), with a of degree 1 such that $\lambda(L) = (5/2)n!$). In any case, the fact that the condition $m_{L_x} = 1$ for all $0 \geq x \in \mathbb{Q}$ implies an inequality sharper than (1.2) is a key step for proving the characterization of the triples on the second Clifford–Severi line.

We are now able to complete the proof of Theorem 1.2:

Proof of Theorem 1.2. (i) Let us define the following real numbers:

$$x_0 := \max\{x \mid \mathrm{vol}_X(L_x) = 0\} \quad \text{and} \quad \bar{x} := \max\{x \mid h^0_a(X, L_x) = 0\}.$$

Since $h^0_a(X, L_x) > 0$ implies that L_x is big, we have that $\bar{x} \geq x_0$. Consider the function

$$\nu(x) := \mathrm{vol}_X(L_x) - 2n!h^0_a(X, L_x).$$

We are going to prove that $\nu(x)$ is identically zero for $x \leq 0$. We have $\nu(0) = 0$ by assumption and $\nu(x) \geq 0$ for $x \leq 0$ by [5, Theorem 6.7]. Hence, it suffices to show that the left derivative $D^-\nu(x)$ is ≥ 0 for $x < 0$.

Using the formulae (2.4) and (2.6) for the left derivatives given in Section 2, we have

$$D^-\nu(x)=\begin{cases}0 & x<x_0\\ n\operatorname{vol}_{X|M}(L_x)-2n!\lim_{d\to\infty}\frac{1}{d^{2q-2}}h^0_{a_d}(X^{(d)}_{|M_d},(L_x)^{(d)}) & x>x_0\end{cases}$$

Observe that for $0\geq x\in\mathbb{Q}$ the inequality:

$$\operatorname{vol}_{X^{(d)}|M_d}\left((P_x)^{(d)}\right)\geq 2(n-1)!h^0_{a_d}(X^{(d)}_{|M_d},(P_x)^{(d)}))$$

holds for $d\gg 0$ by [5, Theorem 6.7], since we can show that $K_{X^{(d)}}-(P_x)^{(d)}$ is pseudoeffective arguing as in the proof of Proposition 3.2. So for rational $0\geq x>x_0$ Remark 2.1 gives:

$$D^-\nu(x)=n(\psi'(x)-2(n-1)!\phi'(x))\geq 0.$$

Since ψ' is continuous for $x\neq x_0$ and ϕ' is non decreasing, the above inequality actually holds for all $x\leq 0$.

We have thus proven that for all $x\leq 0$ one has:

$$\operatorname{vol}_X(L_x)=2\,n!\,h^0_a(X,L_x).$$

Now we can apply Theorem 6.8 of [5] and deduce that for all $x\in\mathbb{Q}\cap(\bar{x},0]$ we have $m_{L_x}=1$ or 2. Indeed, if L_x is integral, this follows directly. Otherwise take t such that $L_x^{(t)}$ is integral. Since the volume and the continuous rank are multiplicative, the same inequality holds for $L_x^{(t)}$ and so we have that $m_{L_x}=m_{L_x^{(t)}}=1$ or 2.

Assume that for all rational $x\leq 0$ with $h^0_a(X,L_x)>0$, we have that $m_{L_x}=1$. Then by Proposition 3.2 we would have that $\operatorname{vol}_X(L)\geq\frac{5}{2}\,n!\,h^0_a(X,L)$, a contradiction. So there exists a rational $t_0=e_0/d_0^2\leq 0$ such that $m_{L_{t_0}}=2$.

As in § 2.5, up to passing to $X^{(d)}$ for $d\gg 0$, taking a suitable blow-up of $X^{(d)}$ and tensoring by α general, we can assume that L_{t_0} is integral and $|L_{t_0}|=|P|+D$, where D is effective, $|P|$ is base point free and $h^0_a(X,L_{t_0})=h^0_a(X,P)$. Moreover, we may assume that the map $\varphi\colon X\to Z$ induced by $|P|$ is the eventual map (see § 2.4) and has degree $m_{L_{t_0}}=2$. Finally, up to replacing both X and Z by modifications, we can assume that Z is smooth, so that there is a morphism $a'\colon Z\to A$ with $a=a'\circ\varphi$. Let $N\in\operatorname{Pic}(Z)$ be such that $P=\varphi^*(N)$; we have $h^0_a(X,P)=h^0_{a'}(Z,N)$. Hence we have that

$$\begin{aligned}2n!h^0_{a'}(Z,N)&=2n!h^0_a(X,L_{t_0})=\operatorname{vol}_X(L_{t_0})\geq\\ &\geq\operatorname{vol}_X(P)=2\operatorname{vol}_Z(N)\geq 2n!h^0_{a'}(Z,N),\end{aligned}$$

where the last inequality follows by [5, Theorem 6.7] (cf. (1.1)). So $\operatorname{vol}_Z(N)=n!h^0_{a'}(Z,N)$ and we can apply Theorem 1.1 to conclude that $q=n$ and that $a'\colon Z\to A$ is birational. Hence $\deg a=2$.

(ii) Let $X \xrightarrow{\eta} \overline{X} \xrightarrow{\bar{a}} A$ be the Stein factorization of a; since $\overline{X}$ is normal and A is smooth, $\bar{a}$ is a flat double cover and, in particular, $\overline{X}$ is Gorenstein and $K_{\overline{X}} = \bar{a}^*N$ for some line bundle N on A. It follows that $K_X = a^*N + E$, where E is η-exceptional. Note that N is ample, since otherwise K_X would not be big.

The usual projection formulae for double covers give $h^0_{\bar{a}}(\overline{X}, K_{\overline{X}}) = h^0_{\mathrm{Id}}(A, N)$, hence for general $\alpha \in \mathrm{Pic}^0(A)$ the paracanonical system $|K_{\overline{X}} \otimes \alpha|$ is a pull back from A. Since the moving part of $|K_X \otimes \alpha|$ is a subsystem of $\eta^*|K_{\overline{X}} \otimes \alpha|$, the map given by $|K_X \otimes \alpha|$ is composed with a (recall that $h^0_a(X, K_X) = \chi(K_X)$ by generic vanishing, so the paracanonical systems $|K_X \otimes \alpha|$ are non empty by assumption).

One can argue exactly in the same way for the degree 2 map $a_d\colon X^{(d)} \to A$ and show that for α general the map given by $|K_{X^{(d)}} \otimes \alpha|$ is composed with a_d. It follows that $m_{K_X} = 2$ and the eventual map of K_X is birationally equivalent to a.

(iii) Under this hypothesis, since $h^0_a(X, L) > 0$, we have $\chi(K_X) = h^0_a(X, K_X) > 0$. As we have shown in (i), there is $0 > t_0 \in \mathbb{Q}$ such that $h^0_a(X, L_{t_0}) > 0$ and $m_{L_{t_0}} = 2$. Up to passing to an étale cover induced by a multiplication map, we have an inclusion $L_{t_0} \to L$ and by assumption we also have an inclusion $L \to K_X \otimes \alpha$, for some $\alpha \in \mathrm{Pic}^0(\mathrm{A})$. Since by (ii) $m_{L_{t_0}} = m_{K_X} = 2$, we can conclude that $m_L = 2$ and that a is also birationally equivalent to the eventual map of L.

□

Remark 3.4. From the above argument it follows that for any $x \leq 0$ the limit for $d \to \infty$ of

$$\frac{1}{d^{2q-2}}\left(\mathrm{vol}_{X^{(d)}|M_d}\left((P_x)^{(d)}\right) - 2(n-1)!h^0_{a_d}(X^{(d)}_{|M_d}, (P_x)^{(d)})\right) \tag{3.6}$$

is zero. However, notice that a posteriori we can observe that for any d the quantity (3.6) is strictly positive. Indeed, if for some $x \in \mathbb{Q}$ we had

$$\mathrm{vol}_{X^{(d)}|M_d}\left((P_x)^{(d)}\right) = 2(n-1)!h^0_{a_d}(X^{(d)}_{|M_d}, (P_x)^{(d)}) \leq 2(n-1)!h^0_{a_d}(M_d, (P_x)^{(d)}{}_{|M_d}),$$

then by the second Clifford Severi inequality we would have equality and by Theorem 1.2 on the triple $(M_d, (P_x^{(d)})_{|M_d}, (a_d)_{|M_d})$ we would have $q = \dim(M_d) = n-1$, a contradiction.

Proof of Corollary 1.4. Given a desingularization $X' \to X$, we have $\mathrm{vol}_{X'}(K_{X'}) = \mathrm{vol}_X(K_X) = K_X^n$. Since the singularities of X are rational and ω_X is the dualizing sheaf of X, we also have that $h^0_{a'}(X', K_{X'}) = \chi(\omega_{X'}) = \chi(\omega_X)$ and the result follows directly from Theorem 1.2. □

Finally we consider the case of curves on the second Severi line. In contrast to the case $n \geq 2$, here the map $a\colon X \to A$ is not always a double cover:

Theorem 3.5. *Let C be a smooth curve of genus $g \geq 4$, let $a\colon C \to A$ be a strongly generating map to an abelian variety and let $L \in \mathrm{Pic}(C)$ be a line bundle with $h^0_a(C, L) \geq 2$. If $\deg L \leq 2g-2$ and $\deg L = 2h^0_a(C, L)$, then one of the following cases occurs:*

(i) *A is an elliptic curve, the map a has degree 2, $L = a^*N$ for some line bundle N on A and $|L \otimes \alpha|$ is not birational for general $\alpha \in \mathrm{Pic}^0(A)$;*
(ii) *$\deg L = 2g-2$ and $|L \otimes \alpha|$ is birational for general $\alpha \in \mathrm{Pic}^0(A)$.*

Proof. Up to replacing L by $L \otimes \alpha$ for some general $\alpha \in \mathrm{Pic}^0(A)$, we may assume that $h^0(C, L) = h^0_a(C, L) =: r + 1$. Plugging the relation $\deg L = 2h^0(C, L)$ into the Riemann-Roch formula we get $h^0(C, L) + h^0(C, K_C - L) = g - 1$, hence in particular $r \leq g - 2$.

Assume $r < g - 2$. In this case we can apply an inequality of Debarre and Fahlaoui [8, Proposition 3.3] on the dimension of the abelian varieties contained in the Brill-Noether locus $W^r_d(C)$, that in our case gives $\dim A \leq (d-2r)/2 = 1$. The Brill-Noether locus $W^r_{2r+2}(C)$ is a proper subset of $J^{2r+2}(C)$ hence by [7, Cor. 3.9] either we are in case (i), or $a\colon C \to A$ has degree 2, C is hyperelliptic and $L = r\Delta + a^*P$, where Δ is the hyperelliptic g^1_2 and $P \in A$ is a point. To exclude the latter case, we claim that a hyperelliptic and bielliptic curve has always genus $g \leq 3$.

Indeed, since the hyperelliptic involution commutes with all the automorphisms, the composition of the bielliptic and the hyperelliptic involutions gives a third involution on C. The first two involutions act on $H^0(C, \omega_C)$ with invariant subspaces of eigenvalue -1 of dimensions g and $g-1$ respectively. So the composition induces a double cover from C to a curve of genus $g-1$. By the Hurwitz formula we obtain the bound $g \leq 3$.

So we have $L = a^*N$, where N is a line bundle of degree $r+1$ on A and $L \otimes \alpha = a^*(N \otimes \alpha)$ for every $\alpha \in \mathrm{Pic}^0(A)$; since $h^0(A, N \otimes \alpha) = r+1 = h^0_a(C, L)$, it follows that $|L \otimes \alpha|$ is not birational for general α.

Assume now $r = g-2$, namely $\deg L = 2g-2$. If $|L \otimes \alpha|$ is birational for general α in $\mathrm{Pic}^0(A)$, then we have case (ii). So assume that $|L \otimes \alpha|$ is not birational for general α in $\mathrm{Pic}^0(A)$. Since $g > 3$, by the Clifford+ Theorem [1, III.3. ex.B.7] the induced map $\phi_{|L\otimes\alpha|}$ factorizes as a double cover σ of a curve D_α of genus 0 or 1, composed with a birational map ϕ_{M_α} such that the moving part of $|L \otimes \alpha|$ is $\sigma^*|M_\alpha|$. Since the group $\mathrm{Aut}(C)$ is finite, the double cover $\sigma\colon C \to D_\alpha$ is independent of $\alpha \in \mathrm{Pic}^0(A)$ general and we may write $D = D_\alpha$.

If $g(D) = 0$, then C is hyperelliptic, σ is the hyperelliptic double cover and $L = r\Delta + F_\alpha$, where Δ is the g^1_2 and F_α is an effective divisor of degree 2. Sending α to F_α defines a generically injective rational map $f\colon A \to W_2(C)$. By [7, Cor. 3.9] we have, as before, $\dim A = 1$, and there is a degree 2 map $\beta\colon C \to A$

such that $f = \beta^*$. So C is both bielliptic and hyperelliptic, hence by the argument previously used it has genus at most 3, against our assumptions.

If $g(D) = 1$, then $L \otimes \alpha$ is a pull-back from D. Since a is strongly generating we have that $A = D$ and hence in particular $\deg a = 2$ and $\dim A = 1$. □

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